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SCALAR CURVATURE OF COMPLEX SUBMANIFOLDS OF A COMPLEX PROJECTIVE SPACE

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1. Statement of results

Let $P_{n+p}(\mathbf{C})$ be a complex projective space of complex dimension $n + p$ with the Fubini-Study metric of constant holomorphic sectional curvature 1, and X be an n-dimensional compact complex submanifold of $P_{n+p}(\mathbf{C})$ with the induced Kaehler structure. Then X is algebraic by a well known theorem of Chow. Throughout this paper, we assume that X is a complete intersection of p hypersurfaces in general position in $P_{n+p}(\mathbf{C})$, i.e., that there exist p hypersurfaces $X_1, \cdots, X_p$ of degree $a_1, \cdots, a_p$ in $P_{n+p}(\mathbf{C})$ such that $X = X_1 \cap \cdots \cap X_p$. As a matter of course, every compact complex hypersurface in $P_{n+p}(\mathbf{C})$ is under consideration. The purpose of the present paper is to prove the following results:

Theorem. *Let X be a complete intersection of p hypersurfaces of degree $a_1, \cdots, a_p$ in general position in $P_{n+p}(\mathbf{C})$, and ρ be the scalar curvature of X. Then*

$$\int_X \rho * 1 = n\{n + p + 1 - (a_1 + \cdots + a_p)\} \int_X * 1 \ ,$$

where $* 1$ denotes the volume element of X.

This theorem implies that the average of the scalar curvature depends only on the degree of X, while the scalar curvature itself on the equations defining X.

Corollary 1. *If $\rho > n^2$ everywhere on X, then $X = P_n(\mathbf{C})$.*

Corollary 2. *Let X be a hypersurface of $P_{n+1}(\mathbf{C})$. If $n(n - \nu + 1) < \rho \leq n(n - \nu + 2)$ everywhere on X, then X is an algebraic manifold of degree ν.*

Let S be the square of the length of the second fundamental form of the imbedding so that $S = n(n + 1) - \rho$. The following corollaries are equivalent to Corollary 1 and Corollary 2 respectively.

Corollary 1′. *If $S < n$ everywhere on X, then $X = P_n(\mathbf{C})$.*

Corollary 2′. *Let X be a hypersurface of $P_{n+1}(\mathbf{C})$. If $n(\nu - 1) \leq S < n\nu$ everywhere on X, then X is an algebraic manifold of degree ν.*

In a previous paper [3], we have proved that *if $S < (n + 2)/(4 - 1/p)$ everywhere on X, then $X = P_n(\mathbf{C})$.* Corollary 1′ is an improvement of this result and is best possible for the following reason: Let $Q_n(\mathbf{C}) = \{(z_0, \cdots, z_{n+1}) \in P_{n+1}(\mathbf{C}) \mid \sum z_i^2 = 0\}$, where $z_0, \cdots, z_{n+1}$ be the homogeneous coordinates

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in $P_{n+1}(\mathbf{C})$. Then $Q_n(\mathbf{C})$ is an Einstein-Kaehler manifold and $S = n$ everywhere on it.

2. Proof of results

Let $g = 2\sum g_{\alpha\beta}dz_\alpha d\bar{z}_\beta$ and $\Phi = \frac{2}{\sqrt{-1}}\sum g_{\alpha\beta}dz_\alpha \wedge d\bar{z}_\beta$ be the Kaehler metric and the fundamental 2-form of X respectively, and let $\mathrm{Ric} = 2\sum R_{\alpha\beta}dz_\alpha d\bar{z}_\beta$ be the Ricci tensor of X. Then the first Chern class $c_1(X)$ of X is represented by the closed 2-form

$$\gamma = \frac{1}{2\pi\sqrt{-1}}\sum R_{\alpha\beta}dz_\alpha \wedge d\bar{z}_\beta .$$

We designate $[\Phi]$ and $[\gamma]$ to be the cohomology classes represented by Φ and γ respectively, so that $c_1(X) = [\gamma]$.

Let h be the generator of $H^2(P_{n+p}(\mathbf{C}), \mathbf{Z})$ corresponding to the divisor class of a hyperplane $P_{n+p-1}(\mathbf{C})$. Then the first Chern class $c_1(P_{n+p}(\mathbf{C}))$ of $P_{n+p}(\mathbf{C})$ is given by

$$c_1(P_{n+p}(\mathbf{C})) = (n + p + 1)h .$$

Let $j\colon X \to P_{n+p}(\mathbf{C})$ be the imbedding, and $\tilde{h}$ the image of h under the homomorphism $j^*\colon H^2(P_{n+p}(\mathbf{C}), \mathbf{Z}) \to H^2(X, \mathbf{Z})$. Then we have

$$c_1(X) = \{n + p + 1 - (a_1 + \cdots + a_p)\}\tilde{h} .$$

Let Ψ be the fundamental 2-form of $P_{n+p}(\mathbf{C})$ so that

$$c_1(P_{n+p}(\mathbf{C})) = \frac{n + p + 1}{8\pi}[\Psi] .$$

These, together with the fact that $\Phi = j^*\Psi$, imply

$$[\Phi] = 8\pi\tilde{h}$$

so that

$$c_1(X) = \frac{1}{8\pi}\{(n + p + 1 - (a_1 + \cdots + a_p))\}[\Phi] .$$

Thus there exists a 1-form η such that

$$\gamma = \frac{1}{8\pi}\{(n + p + 1 - (a_1 + \cdots + a_p))\}[\Phi] + d\eta . \tag{1}$$

Let δ, Λ and M be the usual operators in harmonic integral theory (cf. [2]). Operating Λ on both sides of (1) we have

$$(2) \qquad -\rho/(2\pi) = -n\{n + p + 1 - (a_1 + \cdots + a_p)\}/(2\pi) + \Lambda d\eta ,$$

since $\Lambda\Phi = *(\Phi \wedge *\Phi) = -4n$ and $\Lambda\gamma = *(\Phi \wedge *\gamma) = -\rho/(2\pi)$. On the other hand, using the identity $d\Lambda - \Lambda d = \delta M - M\delta$ and the relation $d\Lambda\eta = M\delta\eta = 0$ we obtain

$$\Lambda d\eta = -\delta M\eta ,$$

and therefore, by (2),

$$(3) \qquad \rho/(2\pi) = n\{n + p + 1 - (a_1 + \cdots + a_p)\}/(2\pi) + \delta M\eta .$$

Integration of both sides of (3) on X thus gives

$$(4) \quad \frac{1}{2\pi}\int_X \rho * 1 = \frac{n}{2\pi}\{n + p + 1 - (a_1 + \cdots + a_p)\}\int_X * 1 + \int_X \delta M\eta * 1 .$$

The second term of the right hand side of (4) vanishes since $\int_X \delta M\eta * 1 = (\delta M\eta, 1) = (M\eta, d1) = 0$, where (,) denotes the global scalar product. Hence we have

$$\int_X \rho * 1 = n\{n + p + 1 - (a_1 + \cdots + a_p)\}\int_X * 1 ,$$

which proves our theorem.

If $n^2 < \rho$ everywhere on X, then

$$n^2 \int_X * 1 < n\{n + p + 1 - (a_1 + \cdots + a_p)\}\int_X * 1 ,$$

which implies $a_1 + \cdots + a_p < p + 1$, that is, $a_1 = \cdots = a_p = 1$, proving Corollary 1.

To prove Corollary 2, we put $p = 1$ and $a_1 = a$. If $n(n - \nu + 1) < \rho \leq n(n - \nu + 2)$ everywhere on X, then

$$n(n - \nu + 1)\int_X * 1 < n(n - a + 2)\int_X * 1 \leq n(n - \nu + 2)\int_X * 1 ,$$

which implies $\nu \leq a < \nu + 1$, that is, $a = \nu$. Hence Corollary 2 is proved.

Bibliography

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